COMBINATORICA 2(3) (1982) 247—274

BRICK DECOMPOSITIONS AND THE MATCHING RANK OF GRAPHS

J. EDMONDS, L. LOVÁSZ and W. R. PULLEYBLANK

Dedicated to Tibor Gallai on his seventieth birthday

Received 1 June 1982

The number of linearly independent perfect matchings of a graph — or, equivalently, the dimension of the perfect matching polytope — is determined in various senses. First it is shown that the fact that every linear objective function can be optimized over the perfect matchings in polynomial time implies the existence of a polynomial-time algorithm to determine the dimension of this set. This observation also yields polynomial algorithms to determine, among others, the number of linearly independent common bases of two matroids and the number of linearly independent maximum stable sets in claw-free or perfect graphs. For the case of perfect matchings, Naddef's minimax theorem for the dimension of the perfect matching polytope is strengthened and it is shown how the decomposition t! eory of matchings in graphs can be applied to derive a particularly simple formula for this dimension. This formula is based upon the number of constituents of a certain decomposition of the graph which we call a brick decomposition. Finally, these results are applied to obtain a description of the facets of the perfect matching polytope.

0. Introduction and notation

Let $G=(V, E)$ be a graph without loops (but multiple edges are allowed). The structure of perfect matchings in G has been the subject of many papers. One approach is to consider the incidence vectors of perfect matchings and their convex hull, which we call the *perfect matching polytope* of G and denote by $PM(G)$. The perfect matching polytope was described as the solution set of a linear system by Edmonds [1]. Two further questions concerning this polytope, however, have remained open: to determine its dimension, and to describe a minimal set of inequalities determining $PM(G)$. For the related matching polytope, the convex hull of the incidence vectors of all (not necessarily perfect) matchings, a minimum set of inequalities was furnished by Pulleyblank and Edmonds [15]. The dimension question does not occur there since the matching polytope is full-dimensional. Naddef [11] gave a minimax formula for the maximum number of linearly independent incidence vectors of perfect matchings in a graph (equivalently, for the dimension of the perfect matching polytope), in terms of the numbers of edges, nodes, and the maximum size

AMS subject classification (1980): 05 C 35; 90 C 05, 05 B 35.

of a family of "strict" cuts with certain additional properties (see Section 1). However, no efficient algorithm was known to compute this number.

In this paper we first show that, using Edmonds' algorithm for finding a maximum weight perfect matching in a general edge-weighted graph as a subroutine, we can easily compute the dimension of $PM(G)$. In fact, the same procedure determines the dimension of any set in polynomial time, provided we have an efficient subroutine to maximize an arbitrary linear objective function over the set. Thus the same procedure gives us the dimension of many other interesting combinatorially defined polyhedra, such as the convex hull of common bases of two matroids, and of maximum independent sets in claw-free or perfect graphs. We feel that a combinatorial study of the dimension of these polytopes, similar to the one carried out in this paper for the perfect matching polytope, should lead to interesting results on the structure of these objects.

A different approach to the study of perfect matchings is a decomposition theory initiated by Kotzig [7] and developed further by Lovász [8] and Lovász and Plummer [9]. This decomposition procedure breaks up an arbitrary graph into 3-connected bicritical graphs, which we call *bricks.* (A graph is *bicritical* if whenever any two nodes are deleted the remaining graph has a perfect matching.) The main result of this paper shows an intimate connection between this decomposition procedure and the dimension of $PM(G)$: if every edge of a connected graph $G=(V, E)$ belongs to a perfect matching, and if the number of bricks obtained by the decomposition procedure is β, then $\dim PM(G)=|E|-|V|+1-\beta$. The special case in which G itself is 3-connected bicritical is in fact the most difficult part of the result. This special case has the following purely graph-theoretic formulation (via Naddef's theorem): if $G=(V, E)$ is a 3-connected bicritical graph and $S\subseteq V$ is an odd set such that $3\leqq|S|\leqq|V|-3$, then there exists a perfect matching in G containing more than one edge connecting S to $V\setminus S$. Even though this lemma sounds very elementary, our proof makes essential use of the linear description of the perfect matching polytope.

Once the dimension of $PM(G)$ is known, it is the matter of a not too difficult but somewhat lengthy computation to describe which defining inequalities give facets of $PM(G)$, and which of them yield the same facet. (Since $PM(G)$ is not full-dimensional, the inequality defining a facet is not unique).

The result on the dimension of $PM(G)$ can also be applied to prove a conjecture of Lovász and Plummer [9] concerning the number of perfect matchings in a bicritical graph: in [9] it was shown that every bicritical graph with n nodes has at least $n/4+2$ perfect matchings, and it was conjectured that this lower bound could be improved to $n/2+1$. This was shown for the case of cubic bicritical graphs by Naddef and Pulleyblank [12] and for Halin graphs, a special case of 3-connected bicritical graphs, by Pulleyblank [14]. We prove this conjecture in general by showing that every bicritical graph with n nodes has at least $n/2+1$ linearly independent incidence vectors of perfect matchings.

The remainder of this section is spent on notation, definitions and some preliminary remarks. In Section 1 we present some lemmas which are basically reformulations of Edmonds' results on the perfect matching polytope. In Section 2 we survey the decomposition procedure mentioned above. In Section 3 we describe the general algorithm for finding the dimension of a set provided an optimization subroutine for linear objective functions is given. Turning our attention to matchings, in Section

4 we study strict cuts, proving in particular the non-existence of non-trivial strict cuts in 3-connected bicritical graphs. Section 5 contains the main formula for the dimension of $PM(G)$, and some special cases and applications. Finally, in Section 6 we discuss the facet-inducing inequalities for the perfect matching polytope.

We adopt the following notation. For a graph $G=(V, E)$ and any $S\subseteq V$, we let $\delta(S)$ denote the set of edges with exactly one end in S and let $\gamma(S)$ denote the set of edges with two ends in S. For any $v\in V$, we abbreviate $\delta(\{v\})$ by $\delta(v)$. For any $S\subseteq V$, we let $G[S]$ denote the node-induced subgraph of G induced by S, i.e., $G[S]=(S, \gamma(S))$. We let $G\setminus S$ denote the graph obtained by deleting all nodes in S and all edges incident with these nodes. Thus $G\setminus S=G[V\setminus S]$. Again, for any $v\in V$, we abbreviate $G\setminus\{v\}$ by $G\setminus v$. For any $S\subseteq V$ and any $v\in V\setminus S$, we let $S+v$ denote $S\cup\{v\}$. Similarly, for any $J\subseteq E$ we let $G\setminus J$ denote the graph $(V, E\setminus J)$ and abbreviate $G\setminus\{j\}$ by $G\setminus j$ for all $j\in E$. If $S\subseteq V$ then we denote by $G\times S$ the graph obtained from G by shrinking S to a single node. With a mild abuse of words, we shall identify the edges of $G\times S$ with some edges of G and for $J\subseteq E$ we denote by $J\times S$ the set of edges of $G\times S$ which are images of edges in J under the shrinking of S.

An *articulation set* in a connected graph $G=(V, E)$ is a set $S\subseteq V$ such that $G\setminus S$ is not connected. Every articulation set S of a k-connected graph contains at least k nodes. Let S be an articulation set of G and let W be the node set of one component of $G\setminus S$. Then $G[W\cup S]\setminus\gamma(S)$ is called a *lobe* of S, and $G[W\cup S]\times S$ a *reduced lobe* of S.

A *cut* in $G=(V, E)$ is a set of the form $J=\delta(S)$, where $\emptyset\subset S\subset V$. The sets S and $V\setminus S$ are called the *shores* of J. If G is connected then every cut determines its shores. If $|V|$ is even then the parities of both shores are the same. We call the cut *odd* or *even* depending on these parities. If both shores of a cut have at least two nodes we call the cut *non-trivial;* a cut of the form $\delta(v)$, $v\in V$ is called a *trivial cut* or a *star*. Cuts $\delta(S_1)$ and $\delta(S_2)$ are called *crossing* if $S_1\not\subset S_2$, $S_2\not\subset S_1$, $S_1\cap S_2\neq\emptyset$ and $S_1\cup S_2\neq V$.

A family of mutually non-crossing cuts will be called *laminar*.

Any perfect matching of G will use an odd number of edges from any odd cut of G. We say that J is a *strict cut* of G if every perfect matching of G uses exactly one edge of J.

We say that $G=(V, E)$ is *critical* if $G\setminus v$ has a perfect matching for every $v\in V$; then $|V|$ is necessarily odd and G is connected.

Recall that G is *bicritical* if $E\neq\emptyset$ and $G\setminus u\setminus v$ has a perfect matching for every $u, v\in V$, $u\neq v$. Clearly this is equivalent to the condition that $E\neq\emptyset$ and $G\setminus v$ is critical for every $v\in V$. If G is bicritical then $|V|$ is even, G has a perfect matching itself, and if $G\neq K_2$ then G is 2-connected. Further, if $\{u, v\}$ is any articulation set then every component of $G\setminus u\setminus v$ has a perfect matching, and so every lobe of $\{u, v\}$ has even cardinality. This implies that each of u and v must have degree at least 4.

The graph $G=(V, E)$ will be called *matching-covered* if $|V|>1$, G is connected and every edge of G occurs in a perfect matching. If $G\neq K_2$ then it follows that G is 2-connected, and so every node in G has degree at least 2.

Finally, for any $x\in\mathbf{R}^E$, for any $J\subseteq E$ we let $x(J)=\sum_{j\in J}x_j$.

1. Preliminaries: the perfect matching polytope

Let $G=(V, E)$ be a graph. With every perfect matching M of G associate its incidence vector $(x_j: j\in E)\in \mathbf{R}^E$ where $x_j=1$ if $j\in M$ and $x_j=0$ otherwise. The convex hull of incidence vectors of perfect matchings (or, briefly, of perfect matching vectors) is denoted by $PM(G)$, and is called the *perfect matching polytope* of G. Edmonds [1] determined a linear system whose solution set is $PM(G)$.

Theorem 1.1 (Edmonds [1]). *Let $G=(V, E)$ be a graph with $|V|$ even. Then $PM(G)$ is the set of vectors $x\in \mathbf{R}^E$ satisfying*

(1.1) $\qquad x_j \geqq 0 \quad$ *for all* $\quad j\in E,$

(1.2) $\qquad x(\delta(i)) = 1 \quad$ *for all* $\quad i\in V,$

(1.3) $\qquad x(J) \geqq 1 \quad$ *for every non-trivial odd cut* J *of* G.

Edmonds proved this theorem by giving an efficient algorithm to maximize a linear objective function cx subject to (1.1), (1.2) and (1.3), for an arbitrary edge weight vector $c\in \mathbf{R}^E$. This algorithm always obtains an integer optimum solution, which proves the theorem. It also constructs an optimum solution to the dual linear program, with half-integral entries, but we shall not need this fact in this paper.

Since $PM(G)$ is not full-dimensional, it is a natural next question to study its dimension.

The first result in this direction is due to Naddef [11], who proved the following. Let $\mathscr{K}$ be a laminar set of odd cuts of a matching-covered graph $G=(V, E)$. Let $J_0\in\mathscr{K}$ and let C_0 be a shore of J_0. Every $J\in\mathscr{K}\setminus J_0$ has a unique shore C_J such that $C_J\subseteqq C_0$ or $C_J\subseteqq V\setminus C_0$. Shrink C_0 and each of these C_J's to form pseudonodes. The resulting graph is called a *$\mathscr{K}$-contraction* of G. We say that $\mathscr{K}$ has the *odd cycle property* if every $\mathscr{K}$-contraction of G contains an odd cycle, i.e., is nonbipartite.

Theorem 1.2. (Naddef [11]). *Let $G=(V, E)$ be a matching covered graph.*

(1.4) *If G is bipartite then*

$$\dim P(G) = |E|-|V|+1.$$

(1.5) *If G is nonbipartite then*

$$\dim PM(G)=|E|-|V|+1-|\mathscr{K}|,$$

where $\mathscr{K}$ is any maximal laminar set of strict nontrivial odd cuts of G which has the odd cycle property.

We call a set $\mathscr{K}$ as in (1.5) a *rank set* of G, since it effectively determines the matching rank. In fact, if $\mathscr{K}$ is a laminar family of strict cuts of G with the odd cycle property then it is not difficult to show that the set of equations $\{x(J)=1\colon J\in\mathscr{K}\}$ together with all degree constraints (1.2) is a set of linearly independent equations. (See [11].) The hard part of Theorem 1.2 is the assertion, for a nonbipartite matching covered graph G, that such a $\mathscr{K}$ can be found satisfying $|\mathscr{K}|=|E|-|V|+1-\dim PM(G)$.

2. Preliminaries continued: matching decompositions of graphs

The matching algorithm of Edmonds [2] in the "cardinality" case, ends up with a maximum matching, together with a partition $V=P\cup I\cup O$ with various important properties. This same partition of V was discovered independently by Gallai [5], and is often called the *Edmonds—Gallai partition* of V. The main properties of this partition are summarized below.

Theorem 2.1. *Let $G=(V,E)$ be a graph. Then V has a (unique) partition $\{O, I, P\}$ with the following properties:*

(a) O consists of exactly those nodes which are missed by some maximum matching;

(b) I consists of exactly those nodes in $V\setminus O$ which are joined to some node in O by an edge;

(c) $G[P]$ has a perfect matching and every component of $G[O]$ is critical;

(d) for every maximum matching M, $M\cap\gamma(P)$ is a perfect matching in $G[P]$ and for every component $K=(V_0, E_0)$ of $G[O]$, $M\cap E_0$ covers all nodes but one of K.

(e) $G[O]$ has exactly $|I|+|V|-2\nu(G)$ connected components. (Recall that $\nu(G)$, the matching number of G, is the number of edges in a maximum cardinality matching of G.)

What will be important for us, in addition, is the fact that this partition can be determined efficiently, i.e., in polynomial time.

If G has a perfect matching then this theorem does not give any structural information. It is not difficult to show that, at least as far as the number and structure of perfect matchings are concerned, we may restrict our attention to matching-covered graphs.

The following result is due to Kotzig [7] and Lovász [8]:

Theorem 2.2. *Let $G=(V,E)$ be a matching-covered graph. Then V has a (unique) partition $\mathcal{P}(G)=\{S_1, \dots, S_k\}$ with the following properties:*

(a) for any two nodes $x, y\in V$, the graph $G\setminus x\setminus y$ has a perfect matching if and only if x and y belong to different classes of $\mathcal{P}(G)$;

(b) if $S_i\in\mathcal{P}(G)$ and $H_1, \dots, H_m$ are the components of $G\setminus S_i$, then $m=|S_i|$ and every H_j is critical;

(c) shrinking every component H_j to a single point we obtain from G a bipartite matching-covered graph;

(d) the reduced lobes of S_i are matching-covered graphs.

Note that property (a) implies that the partition $\mathcal{P}(G)$ is a colouration of G. It is easily seen that G is bipartite if and only if $|\mathcal{P}(G)|=2$ and G is bicritical if and only if $|\mathcal{P}(G)|=|V|$.

Let us point out that the partition $\mathcal{P}(G)$ can be found efficiently for every matching-covered graph G. The procedure is the following. Select any node $v_1\in V$, and determine the Edmonds—Gallai partition (O_1, I_1, P_1) for the graph $G\setminus v_1$. Let $S_1=I_1\cup\{v_1\}$. Then select any $v_2\in V\setminus S_1$, determine the Edmonds—Gallai partition (O_2, I_2, P_2) of $G\setminus v_2$, and set $S_2=I_2\cup\{v_2\}$, etc.

Let G be a non-bicritical matching-covered graph. Select any $S\in\mathcal{P}(G)$ with at least two nodes, and construct the reduced lobes of S. These are themselves matching-covered graphs by (d). If any of them is non-bicritical then repeat this with this reduced lobe. Eventually we end up with a list $L_1(G)$ of bicritical graphs. In this list, there will be K_2's which will not concern us much; there will be 3-connected bicritical graphs, which cannot be (and, as we shall see, need not be) decomposed any further by similar means; and there will be non-3-connected bicritical graphs with more than 2 nodes. Such bicritical graphs can be decomposed as follows (Lovász and Plummer [9]).

Let $G=(V, E)$ be a bicritical graph and $\{u, v\}$ an articulation set, and let $G_1, \ldots, G_r$ be the lobes of $\{u, v\}$. Then $G_1+uv, \ldots, G_r+uv$ are again bicritical graphs. If one of $G_1+uv, \ldots, G_r+uv$ is not 3-connected we can repeat this. In this way we can decompose every bicritical graph into 3-connected bicritical graphs. If this decomposition procedure is applied to all bicritical graphs in $L_1(G)$ with more than two nodes, we obtain a list $L(G)$ of 3-connected bicritical graphs. We call the members of $L(G)$ the *3-connected bicritical constituents,* or simply the *bricks,* of G. We will refer to this decomposition of a matching covered graph G as a *brick decomposition* of G.

Let us remark that the decomposition procedure is not uniquely determined: we have freedom in choosing $S_i \in \mathcal{P}(G)$ and also in choosing an articulation pair when decomposing bicritical graphs. It will follow from our results in Section 5 that the number of bricks is independent of the choices made in the decomposition. Lovász (unpublished) also proved that in fact the list $L(G)$ is independent of the order of decomposition; but we shall not need this fact in this paper.

Let us emphasize once more that the decomposition procedure described above can be carried out efficiently for every graph G. Also, note that when we perform the brick decomposition, we will end up with a list containing, in general, some bricks and some K_2's. For example, if G is a bipartite matching covered graph, then the brick decomposition decomposes G into a set of K_2's — one for each edge. If G is itself 3-connected and bicritical then G is indecomposable and the list consists of the single brick G itself.

3. Solvable sets and polyhedra

The main result of this section is the following: if we have a class $\mathcal{K}$ of subsets of $\mathbf{R}^n$ such that for every $S \in \mathcal{K}$ and for any $c \in \mathbf{R}^n$ we can compute the maximum of cx for $x \in S$ in polynomial time, then we can compute dim S, the dimension of S, for any $S \in \mathcal{K}$ in polynomial time. First we introduce some definitions and notation which will enable us to formulate the above result precisely.

Let S be a set of rational members of $\mathbf{R}^n$. An *encoding* of S is a pair $(n, \alpha(S))$ where $\alpha(S)$ is some finite binary string that, in some way, represents S. For example, if S is a finite set of rational points in $\mathbf{R}^n$ then $\alpha(S)$ might be a binary encoding of a linear system whose solution set is the convex hull of S. If S is the set of incidence vectors of the matchings of a graph G, then $\alpha(S)$ could be a string representing the adjacency matrix of G. The *length* of $(n, \alpha(S))$ is defined to be $\lfloor \log n \rfloor + 1$ plus the number of bits in $\alpha(S)$.

We say that an encoding $(n, \alpha(S))$ is *proper* if, for every $x \in S$, the total number of bits required to represent x is polynomially bounded in the length of $(n, \alpha(S))$. In fact, most "reasonable" encodings are proper, as are, in particular, the two examples above. However if we encoded the set M_p of all matchings of K_p, the complete graph on p nodes, by simply letting $\alpha(M_p)$ be the binary representation of p, this would not be proper.

Let $\mathscr{K}$ be a class of proper descriptions of rational subsets of $\mathbf{R}^n$. If there exists an algorithm which, for every $(n, \alpha(S))\in\mathscr{K}$ and every integer vector $c=(c_i\colon i=1, 2, \ldots, n)\in\mathbf{R}^n$, computes $x^*\in S$ which maximizes cx over $x\in S$ in time polynomial in the length of $(n, \alpha(S))$ and $\sum_{i=1}^{n}(1+\lfloor\log_2 c_i\rfloor)$ then we say that $\mathscr{K}$ is a *solvable* class of proper descriptions.

Theorem 3.1. *For every solvable class $\mathscr{K}$ of proper descriptions, for every $(n, \alpha(S))\in\mathscr{K}$, the dimension of S can be computed in time polynomial in the length of $(n, \alpha(S))$.*

We prove the theorem by giving a polynomial algorithm for computing $\dim S$ from its description $(n, \alpha(S))$. More precisely, it will produce a set X of $\dim S+1$ affinely independent members of S together with a system $Ax=b$ of $n-\dim S$ affinely independent equations satisfied by every $x\in S$. (We say that the equations $Ax=b$ are affinely independent if the rows of A are affinely independent.)

The procedure for calculating A, b and X proceeds as follows. Initially $X:=\emptyset$ and $Ax=b$ is a set of $n+1$ affinely independent equations having empty solution set. Thus, for every $x\in X$, we have $Ax=b$. We now proceed to "verify" each equation $ax=\beta$ of our system in turn, by first solving $Z=\max\{ax\colon x\in S\}$ and then solving $z=\min\{ax\colon x\in S\}$. If $z=Z=\beta$ then this is indeed an equation satisfied by every member of S so we leave it in the system and go on to the next. If one of z, Z is not equal to β, then one of the optimum solutions x^* obtained will be affinely independent of X and we add it to X. We then remove the equation $ax=\beta$ from our system, but first use it to update all "unverified" inequalities so that we will still have $Ax=b$ for every $x\in X$. When this "verification" process is completed, we will have the desired A, b and X.

Although the idea of this algorithm is quite simple, some care must be exercised in order to obtain the polynomial bound on its execution. The important point will be to show that at each stage of the algorithm the coefficients of A and b will be polynomially bounded in the length of $(n, \alpha(S))$.

In fact, to calculate A, b and X we consider a homogenized version. For any $x\in\mathbf{R}^n$ we let $\tilde{x}\in\mathbf{R}^{n+1}$ be obtained from x by adding an $(n+1)$ st component with value 1. For $X\subseteq\mathbf{R}^n$ we let $\tilde{X}:=\{\tilde{x}\colon x\in X\}$. Every $x\in X$ satisfies $Ax=b$ if and only if every $\tilde{x}\in\tilde{X}$ satisfies $[A, -b]\tilde{x}=0$. Moreover, X is affinely independent if and only if $\tilde{X}$ is linearly independent. We will construct A and b so that the rows of $[A, -b]$ are linearly independent. Provided that S is nonempty, i.e., that some $\hat{x}$ satisfies $A\hat{x}=b$, it will then follow that the rows of A are linearly independent.

For a matrix A we let A_i denote the ith row and we let $A_{i,j}$ denote the element indexed by i and j. Initially, let A^0 be the $(n+1)\times(n+1)$ identity matrix and let $\tilde{X}^0:=\emptyset$. The algorithm computes a sequence $A^0, A^1, \ldots, A^{n+1}$ of matrices and a sequence $\tilde{X}^0, \tilde{X}^1, \ldots, \tilde{X}^{n+1}$ of subsets of $\mathbf{R}^{n+1}$ such that for each $k\in\{0, 1, \ldots, n+1\}$,

(a) the rows of A^k are linearly independent;
(b) the set $\tilde{X}^k$ is linearly independent;
(c) $A^k\tilde{x}=0$ for all $\tilde{x}\in\tilde{X}^k$;
(d) for every $\tilde{x}\in\tilde{X}^k$, $x\in S$;
(e) the matrix A^k contains $v(k)+(n+1)-k$ rows, for some value $v(k)$ and every $x\in S$ satisfies $A_i^k\tilde{x}=0$ for all i satisfying $1\leqq i$ and $i\leqq v(k)$.

The value $v(k)$ can be interpreted as the number of "verified" rows of A^k. We will have $v(n+1)$ equal to the number of rows of A^{n+1}. Define $v(0):=0$ and then $A^0, \tilde{X}^0$ satisfy (a)—(e).

For k going from 0 to n, perform the following:

Let a^k denote the vector consisting of the first n components of $A^k_{v(k)+1}$. Solve the two problems

$$Z = \text{maximum}\,(a^k x:\ x \in S),$$

$$z = \text{minimum}\,(a^k x:\ x \in S) = \text{maximum}\,(-a^k x:\ x \in S).$$

If $Z = z = -A^k_{v(k)+1,n+1}$ then perform Step A; otherwise perform Step B.

Step A: *For every* $x \in S$, $\hat{x}$ *satisfies* $A_{v(k)+1}\tilde{x}=0$. *Let* $A^{k+1}:=A^k$; $\tilde{X}^{k+1}:=\tilde{X}^k$; $v(k+1):=v(k)+1$.

Step B: *There exists* x *satisfying* $a^k x=Z$ *or* $a^k x=z$ *such that* $A^k_{v(k)+1}\tilde{x}=\alpha\neq 0$. *Let* $\tilde{x}^k:=\tilde{x}$ *and let* $\tilde{X}^{k+1}:=\tilde{X}^k\cup\{\tilde{x}^k\}$. *We now compute the matrix* A^{k+1} *as follows. For* $i\leq v(k)$ *we let* $A^{k+1}_i:=A^k_i$. *For* $i>v(k)+1$, *if* $A^k_i\tilde{x}^k=\beta$ *we let*

$$A^{k+1}_{i-1}:=(\alpha\cdot A^k_i-\beta\cdot A^k_{v(k)+1})/\alpha'$$

where $\alpha'=1$ *if this is the first execution of Step B and* α' *is the value of* α *obtained at the previous execution of Step B if this is not the first execution.* (*Note that if* $\beta=0$ *we have* $A^{k+1}_{i-1}=(\alpha/\alpha')A^k_i$) *Moreover,* A^{k+1} *contains one fewer row than* A^k; *row* $v(k)+1$ *has been deleted. Let* $v(k+1):=v(k)$.

The correctness of this procedure can be easily verified, by showing that at the end of each execution of Step A or B, we have (a)—(e) satisfied. Properties (a) and (d) are immediate and (b) follows from the fact that if we add $\tilde{x}^k$ to $\tilde{X}^k$ in Step B, then every $\tilde{x}\in\tilde{X}^k$ satisfies $A^k x=0$, but $A^k\tilde{x}^k\neq 0$. Property (c) holds because if we add $\tilde{x}^k$ to $\tilde{X}^k$, then we calculate A^{k+1} in such a way that it preserves (c). Finally, (e) holds because we only increment $v(k)$ in Step A, in which case we have found another row satisfying $A_{v(k)+1}\tilde{x}=0$ for all $x\in S$.

Now we show that the entire process runs in time polynomial in the length of $(n, \alpha(S))$. It is sufficient to show that every coefficient in each A^k has length polynomially bounded in $(n, \alpha(S))$. This will follow from a theorem of Edmonds [4] which shows that the form of "Gaussian elimination" used in Step B has this desired property.

Let M be the matrix obtained by adjoining $|\tilde{X}^{n+1}|$ new columns to A^0, one new column containing each $\tilde{x}\in\tilde{X}^{n+1}$. Since $\mathcal{K}$ is a class of proper descriptions, the number of bits required to represent M is polynomial in the length of $(n, \alpha(S))$. In [4] it is shown that if "Gaussian elimination" is performed using the method of Step B, then each entry in the resulting matrix will be equal to the determinant of a square submatrix of M and hence the length of each entry will be polynomial in the length of $(n, \alpha(S))$. We choose as pivot elements for this process the kth element of each column corresponding to $\tilde{x}^k\in\tilde{X}^{n+1}$.

Finally, we complete the proof by observing that the matrix A^k of the algorithm will be identical to the submatrix of M obtained by taking the columns corresponding to A^0, after $k-v(k)$ pivots have been performed and all rows pivoted on have been deleted. The entries in the column of M corresponding to $\tilde{x}^k$ give the values of $A_i^k \tilde{x}^k$ for all $i \in \{1, 2, \dots, n+1\}$.

This algorithm can be applied to efficiently determine the dimension of various combinatorial polyhedra for which polynomial algorithms exist to maximize any linear objective function. Another way to say this is that it computes the number of affinely independent vertices of the polytope.

As special cases, we obtain polynomial algorithms for the following.

(a) The number of affinely independent perfect matchings, since there exists a polynomial algorithm to maximize a linear objective function over the perfect matching polytope, by Edmonds [1];
(b) The number of affinely independent common bases of two matroids, using the algorithm of Edmonds [3];
(c) The number of affinely independent maximum cliques of a perfect graph, using the algorithm of Grötschel, Lovász and Schrijver [6];
(d) The number of affinely independent maximum independent sets in a claw-free graph, using the algorithm of Minty [10].

Let P be a polytope and let $ax \leqq b$ be an inequality valid for every $x \in P$. Let

$$P' = \{x \in P | ax = b\}.$$

Then P' is a face of P. Now it may be that $P' = P$, i.e., $ax = b$ for every $x \in P$, in which case we call the inequality *tight*. It may be that P' is a facet of P, in which case we call the inequality $ax \leqq b$ *facet-inducing*. Finally, it may happen that P' is a face but not a facet of P, in which case we call the inequality $ax \leqq b$ *inessential*.

If P is full-dimensional, then every facet of P determines a unique facet-inducing inequality up to a scalar factor. This is not the case when P is not of full dimension. In fact, the inequalities $ax \leqq b$ and $a'x \leqq b'$ determine the same facet of P if and only if there is $\lambda > 0$ and an equation $cx = d$ satisfied by all points in P such that $a' = \lambda a + c$ and $b' = \lambda b + d$.

The significance of this classification of inequalities is that if P is given as the solution set of a linear system $a_i x \leqq b_i$ $(i = 1, \dots, m)$, then any inequality $a_i x \leqq b_i$ which is inessential can be dropped from the system. Further, from those inequalities $a_i x \leqq b_i$ which are tight one may select any maximal subset whose left hand sides are linearly independent and then drop the rest. Finally, for each facet one may drop all but one of the facet-inducing inequalities for that facet.

Let P be a rational polytope in $\mathbf{R}^n$ and let S be the set of vertices of P. Let $(n, \alpha(P))$ be an encoding of P, where $\alpha(P)$ is some finite binary string that represents P. Then $(n, \alpha(P))$ is also an encoding of S and we say that it is a *proper encoding* of P if it is a proper encoding of S. We say that $\mathscr{P}$ is a *solvable class of proper polyhedral descriptions* if $\mathscr{P}$ is a solvable class of descriptions of the vertex sets of the polytopes encoded in $\mathscr{P}$.

Let P be a polytope and let $ax \leqq b$ be an inequality valid for all points of P, such that at least one point of P gives equality. Then the polytope

$$P' = \{x \in P | a \cdot x = b\}$$

is a face of P. If there exists a polynomial algorithm to maximize any linear objective function over P, then there is also one to maximize over P'. In fact, maximizing

$c \cdot x$ over P' is equivalent to maximizing $(c+Na)\cdot x$ over P, for N sufficiently large. All we must ensure is that if we have a proper description of P, then a "sufficiently large" N will still have length polynomially bounded in the length of this description and the sum of the lengths of the coefficients of c, a and b.

Let $(n, \alpha(P))$ be a proper description of P. Then the length of each coefficient of each vertex of P is polynomially bounded in the length of $(n, \alpha(P))$ so there is a polynomially computable upper bound D on the l.c.m. of the coefficients of any vertex. The length of D will also be polynomial in the length of $(n, \alpha(P))$. We can assume that a and b are integer. Then, for any $x \in P-P'$, $ax \leqq b-(1/d)$.

Let $C_{\max}:=\max\{cx : x\in P\}$ and let $C_{\min}:=\min\{cx : x\in P\}$. Then the lengths of $C_{\max}$ and $C_{\min}$ are both polynomial in the length of $(n, \alpha(P))$ and the sum of the lengths of the coefficients of c. Now let $N=1+(C_{\max}-C_{\min})D$. Then the length of N is polynomial as required. For $x'\in P'$ we have $(c+Na)x'=cx'+N\cdot b$. For $x\in P-P'$, we have $(c+Na)x \leqq cx+Nb-N/D<cx+Nb+C_{\min}-C_{\max}\leqq Nb+cx'$. Therefore every $x\in P-P'$ gives a worse value for $(c+Na)x$ than every $x\in P'$, so N is sufficiently large. (Note that if the vertices of P are integer, then $D=1$ so we simply choose $N>C_{\max}-C_{\min}$.)

Theorem 3.2. *Let $\mathscr{P}$ be a solvable class of proper polyhedral descriptions. Then there exists a polynomial algorithm which, given any $(n, \alpha(P))\in\mathscr{P}$ and any inequality $ax\leqq b$ $(a\in \mathbf{Z}^n, b\in\mathbf{Z})$, determines whether $ax\leqq b$ is invalid, tight, facet-inducing or inessential.*

The algorithm can be described as follows. Find an $x_0\in P$ such that $a\cdot x_0=\max\{a\cdot x | x\in P\}$. If $ax_0>b$, then $ax\leqq b$ is invalid; if $ax_0<b$ then $ax\leqq b$ is inessential. So suppose that $ax_0=b$.

Let $P'=\{x\in P, ax=b\}$. Determine $\dim P$ and $\dim P'$ as above. If $\dim P=\dim P'$ then $ax\leqq b$ is tight. If $\dim P=\dim P'+1$, then $ax\leqq b$ is facet-inducing. If $\dim P>\dim P'+1$, then $ax\leqq b$ is inessential.

Theorem 3.3. *Let $\mathscr{P}$ be a solvable class of proper polyhedral descriptions. Then there exists a polynomial algorithm which, given any $(n, \alpha(P))\in\mathscr{P}$ and two facet-inducing inequalities $ax\leqq b$ and $a'x\leqq b'$, determines whether or not they induce the same facet.*

In fact, it suffices to check whether or not $a'x\leqq b$ is tight for the polytope $P'=\{x\in P: ax=b\}$.

4. Strict cuts

Recall that an odd cut of a graph G is *strict* if every perfect matching contains exactly one edge. Two examples of strict cuts may be instructive at this point. First, let G be a matching-covered non-bipartite non-bicritical graph, $S\in\mathscr{P}(G)$, $|S|\geqq 2$ and let H be a connected component of $G\setminus S$ with more than one node. Then $\delta(V(H))$ is a non-trivial strict cut. Second, let G be a matching-covered graph which is not 3-connected and let $\{u, v\}$ be an articulation set of G. Let H be an even connected component of $G\setminus\{u, v\}$. Then $\delta(V(H)\cup\{u\})$ is a non-trivial strict cut.

These two types of strict cuts are shown in Figure 1.

Let $G=(V, E)$ be a graph with $|V|$ even and let J be an odd cut. We can test whether or not J is strict by defining edge costs $c_j \equiv 1$ for $j \in J$ and $c_j \equiv 0$ for $j \in E \setminus J$ then finding a maximum weight perfect matching in G. Then J is strict if and only if the optimum matching found contains exactly one edge of J. Now we show how we can use Theorem 1.1 for this problem to obtain several useful lemmas.

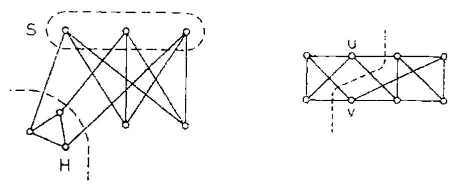

Fig. 1

Let $\mathscr{C}$ be the set of all non-trivial odd cuts in G.

An *odd cut family* $\mathscr{K}=(J_i : i \in I)$ is a family of odd cuts of G. (Note that the same cut may appear several times.) If every member of $\mathscr{K}$ is a star, then we call $\mathscr{K}$ a *star family*. The *cover vector* of $\mathscr{K}$, denoted by $e(\mathscr{K})$, indicates for each edge j of G how many members of $\mathscr{K}$ contain j. Thus

$$e(\mathscr{K})_j \equiv |\{i \in I:\ j \in J_i\}|.$$

If $\mathscr{K}_1$ and $\mathscr{K}_2$ are odd cut families, then we say that $\mathscr{K}_1$ *majorizes* $\mathscr{K}_2$ if $e(\mathscr{K}_1) \geqq e(\mathscr{K}_2)$, i.e., each edge appears at least as often in $\mathscr{K}_1$ as in $\mathscr{K}_2$. We let $|\mathscr{K}|$ denote $|I|$. Thus $|\mathscr{K}|$ counts multiplicities of odd cuts in $\mathscr{K}$.

Lemma 4.1. *An odd cut J_0 is strict if and only if there exists an odd cut family $\mathscr{K}$ containing J_0 and a star family $\mathscr{S}$ such that $\mathscr{S}$ majorizes $\mathscr{K}$, and $|\mathscr{S}|=|\mathscr{K}|$.*

Proof. Let c be the incidence vector of J_0. Then J_0 is strict if and only if the

(4.1) $\qquad$ maximum of $c \cdot x$,

subject to

(4.2) $\qquad x(j) \geqq 0 \quad$ for all $\ j \in E$,

(4.3) $\qquad x(\delta(i)) = 1 \quad$ for all $\ i \in V$,

(4.4) $\qquad x(J) \geqq 1 \quad$ for all $\ J \in \mathscr{C}$,

equals 1. Since this maximum is at least 1 by (4.4), it equals 1 if and only if the dual program of (4.1)—(4.4) has a dual solution with value 1. This means that there exist rationals $y^*(i)$ $(i \in V)$ and $z^*(J)$ $(J \in \mathscr{C})$ such that

(4.5) $\qquad z^*(J) \geqq 0 \quad$ for all $\ J \in \mathscr{C}$,

(4.6) $\qquad y^*(\psi(j)) \geqq \sum (z^*(J):\ j \in J \in \mathscr{C}) + 1 \quad$ for all $\ j \in J_0$,

(4.7) $\qquad y^*(\psi(j)) \geqq \sum (z^*(J):\ j \in J \in \mathscr{C}) \quad$ for all $\ j \in E - J_0$

and

(4.8) $$y^*(V) = z^*(\mathscr{C})+1.$$

Here we let $\psi(j)$ denote the two nodes incident with edge j.

Let q be a common denominator of the entries of y^* and z^*. Define $\mathscr{S}$ by letting each star $\delta(v)$ appear $qy^*(v)$ times if $y^*(v)>0$ .Define $\mathscr{K}$ by letting each star $\delta(v)$ appear $-qy^*(v)$ times if $y^*(v)<0$, letting each non-trivial odd cut $J\in\mathscr{C}\setminus\{J_0\}$ appear $q\cdot z^*(J)$ times, and letting J_0 appear $q(z^*(J_0)+1)$ times. Then $|\mathscr{S}|=|\mathscr{K}|$ by (4.8), $J_0\in\mathscr{K}$ and $\mathscr{S}$ majorizes $\mathscr{K}$ by (4.6) and (4.7). Conversely, if such families $\mathscr{S}$ and $\mathscr{K}$ exist then a solution of (4.5)—(4.8) can be defined in the obvious way. ∎

If G is matching covered and J is any strict cut then every edge belongs to a perfect matching containing exactly one edge of J. Therefore if y^*, z^* is an optimum solution of (4.5)—(4.8) as in the preceding proof, then we must have equality in (4.6) and (4.7) for all $j\in E$ by the complementary slackness principle of linear programming. Thus the odd cut families $\mathscr{K}$ and $\mathscr{S}$ will satisfy $e(\mathscr{K})=e(\mathscr{S})$. Thus we have the following.

Lemma 4.2. *If G is matching-covered and $\mathscr{K}$ is an odd cut family majorized by a star family $\mathscr{S}$ such that $|\mathscr{K}|=|\mathscr{S}|$, then $e(\mathscr{K})=e(\mathscr{S})$.*

We remark that Lemma 4.1 is only interesting for the case that J_0 is a non-trivial odd cut. If J_0 is a star $\delta(v)$ then letting $\mathscr{S}=\mathscr{K}=\{\delta(v)\}$ trivially satisfies the conditions of the lemma.

We can reinterpret the previous lemmas in terms of node weights and edge capacities. Let $\pi=(\pi_i\in\mathbf{R}: i\in V)\geqq 0$ be a node weight vector and define an edge capacity vector $\tilde{\pi}$ by letting $\tilde{\pi}_j=\pi_u+\pi_v$ if $j=uv\in E$. That is, each edge's capacity is the sum of the weights of its two endpoints.

Let π be a non-negative integer node weight vector, and consider the following problem, which we denote by $P(\pi)$: maximize $|\mathscr{K}|$ for odd-cut families $\mathscr{K}$ satisfying $e(\mathscr{K})\leqq\tilde{\pi}$. It is clear that the family $\mathscr{K}_0$ which consists of π_v copies of $\delta(v)$ is a solution and it has $|\mathscr{K}_0|=\pi(V)$. Suppose that G has a perfect matching M. Then we have $\tilde{\pi}(M)=\pi(V)$. Let $\mathscr{K}$ be any solution of $P(\pi)$. Since every odd cut contains at least one edge of M, and $e(\mathscr{K})\leqq\tilde{\pi}$, we have $|\mathscr{K}|\leqq\tilde{\pi}(M)=\pi(V)$. Thus if G has a perfect matching then the optimal solutions of $P(\pi)$ are precisely those odd-cut families $\mathscr{K}$ which satisfy $e(\mathscr{K})\leqq\tilde{\pi}$ and $|\mathscr{K}|=\pi(V)$.

Let G be a graph with a perfect matching, $\mathscr{K}$ a family of odd cuts in G and $\mathscr{S}$ a star family such that $\mathscr{S}$ majorizes $\mathscr{K}$ and $|\mathscr{S}|=|\mathscr{K}|$. For each $i\in V$, define π_i to be the number of times $\delta(i)$ occurs in $\mathscr{S}$. Then $e(\mathscr{K})\leqq\tilde{\pi}$ and $|\mathscr{K}|=\pi(V)$, so $\mathscr{K}$ is an optimum solution of $P(\pi)$. Since this construction can be reversed, we obtain the following equivalent form of Lemma 4.1:

Corollary 4.3. *An odd cut J in a graph G with a perfect matching is strict if and only if there exists a vector π of nonnegative integral node weights such that J occurs in an optimum solution to $P(\pi)$.*

We also have the following version of Lemma 4.2.

Corollary 4.4. *If $G=(V,E)$ is matching-covered and $\mathscr{K}$ is an optimum solution to $P(\pi)$ for some non-negative integer node weight vector π then $e(\mathscr{K})=\tilde{\pi}$.*

The following property of strict cuts enables us to perform some simplifications to strict cut families. The proof is left to the reader.

Proposition 4.5. *Let J_1 and J_2 be strict cuts. Select shores S_1 of J_1 and S_2 of J_2 such that $|S_1 \cap S_2|$ is odd. Then $\delta(S_1 \cap S_2)$ and $\delta(S_1 \cup S_2)$ are also strict cuts.*

We mention one more useful property of strict cuts.

Lemma 4.6. *Let J be a strict cut in a matching-covered graph $G=(V, E)$ and let S be a shore of J. Then S induces a connected graph. If, in addition, G is bicritical and $|S|>1$, then S induces a non-bipartite graph.*

Proof. First we note that $G[S]$ must be connected. For if not, at least one component K of $G[S]$ would have $|V(K)|$ odd. Let H be a different component. Then for any $j \in \delta(V(H))$ we would have $j \in \delta(S)$ so a perfect matching of G that contained j would have to induce a perfect matching of K which is impossible. Therefore H must be a component of G, a contradiction, since matching-covered graphs are connected.

Suppose that G is bicritical, $|S|>1$ and yet $G[S]$ is bipartite with parts S_1 and S_2, with $|S_1|<|S_2|$. Then, since S is strict, we must have $|S_1|=|S_2|-1$ and since G is matching covered every edge of $\delta(S)$ is incident with a node of S_2. But $|S_2| \geqq 2$ and if we delete two nodes of S_2, then clearly no perfect matching of G can exist, contradictory to G being bicritical. ∎

The following theorem is a key result in this paper.

Theorem 4.7. *A 3-connected bicritical graph has no non-trivial strict cut.*

Proof. Suppose, in order to obtain a contradiction, that there exists a 3-connected bicritical graph $G=(V, E)$ which has a non-trivial strict cut J. Since bicritical graphs are matching-covered it follows from Corollary 4.3 that there exists a vector π of non-negative integer node weights and an odd cut family $\mathscr{K}$ containing J which is an optimum solution to $P(\pi)$. First we simplify the structure of $\mathscr{K}$.

Claim 1. *There exists a laminar odd cut family $\mathscr{K}$ containing at least one non-trivial cut which is an optimum solution to $P(\pi)$.* (We do not claim that the non-trivial strict cut in $\mathscr{K}$ is the same cut J we started with).

In fact, choose an optimum solution $\mathscr{K}$ of $P(\pi)$ containing at least one non-trivial cut such that the number of pairs of cuts in $\mathscr{K}$ that cross is minimum. If no cuts cross we are finished. Otherwise let J_1 and J_2 be two cuts that cross and let S_1 and S_2 be shores of J_1 and J_2, respectively, such that $|S_1 \cap S_2|$ is odd. By Proposition 4.5, $\delta(S_1 \cup S_2)$ and $\delta(S_1 \cup S_2)$ are strict cuts. Let $\mathscr{K}'$ be obtained from $\mathscr{K}$ by replacing J_1 and J_2 with the cuts $\delta(S_1 \cap S_2)$ and $\delta(S_1 \cup S_2)$. Then $e(\mathscr{K}') \leqq e(\mathscr{K}) = \tilde{\pi}$ and $|\mathscr{K}'| = |\mathscr{K}|$. Therefore $\mathscr{K}'$ is also an optimal solution to $P(\pi)$, and so by Corollary 4.4, $e(\mathscr{K}') = \tilde{\pi}$. Hence it follows that no edge connects $S_1 \setminus S_2$ to $S_2 \setminus S_1$, since for such an edge j we would have $e(\mathscr{K}')_j = e(\mathscr{K})_j - 2 < \tilde{\pi}_j$, a contradiction. Thus, $(S_1 \cap S_2) \cup (V \setminus (S_1 \cup S_2))$ is an articulation set and so by hypothesis it must contain at least three nodes. But then one of $\delta(S_1 \cap S_2)$ and $\delta(S_1 \cup S_2)$ is not a star, and so $\mathscr{K}'$ contains at least one non-trivial cut. Since $\mathscr{K}'$ contains fewer pairs of cuts that cross than does $\mathscr{K}$, we have a contradiction to our choice of $\mathscr{K}$. This completes the proof of Claim 1.

So let $\mathscr{K}$ be as in Claim 1. Choose π and $\mathscr{K}$ such that $|\mathscr{K}|$ is minimal. Since $\mathscr{K}$ contains at least one non-trivial cut, $\mathscr{K}\neq\emptyset$. Let $J\in\mathscr{K}$ (trivial or non-trivial), $j\in J$, and let $j=uv$. Then $e(\mathscr{K})_j=\tilde{\pi}_j>0$ and so at least one of the endpoints of j has positive weight π. Let u be an endpoint of j such that $\pi_u>0$ and let S be the shore of J containing u. Among all possible choices of J, j and u, pick one for which $|S|$ is minimal.

Claim 2. $|S|>1$.

For assume that $S=\{u\}$. Then omitting $J=\delta(u)$ from $\mathscr{K}$ and decrementing π_u by 1 we get another laminar family $\mathscr{K}'$ still containing a nontrivial cut and another nonnegative integral node weight vector π' such that $\mathscr{K}'$ is an optimum solution to $P(\pi')$. Since $|\mathscr{K}'|<|\mathscr{K}|$, this is a contradiction.

Let $S_1, \dots, S_m$ be the maximal proper subsets of S for which $\delta(S_i)\in\mathscr{K}$. Let $S^+=\{x\in S\setminus(S_1\cup\dots\cup S_m)\colon \pi_x>0\}$ and $S^0=S\setminus(S^+\cup S_1\cup\dots\cup S_m)$.

Claim 3. *No edge of G joins nodes belonging to distinct sets S_i.*

For assume that $k=w\overline{w}$ joins S_1 and S_2 (say). Then $e(\mathscr{K})_k>0$ so $\pi_w>0$ or $\pi_{\overline{w}}>0$. Let, say, $\pi_w>0$, then $\delta(S_1)$, k and w should have been chosen instead of J, j and u.

Claim 4. *No edge connects S^0 to any S_i.*

This follows just as Claim 3.

Claim 5. *No edge connects two nodes of S^+ or a node of S^+ to a node of S^0.*

For suppose that k is such an edge. Then $e(\mathscr{K})_k=0$ but $\tilde{\pi}_k>0$, which contradicts Corollary 4.4.

Since by Lemma 4.6 the subgraph induced by S is connected, we can conclude that $S^0=\emptyset$.

Claim 6. $|S^+|=m+1$.

For consider any perfect matching M containing the edge j. Since each member of $\mathscr{K}$ is strict, each $\delta(S_i)$ contains exactly one edge of M. These m edges have to go to S^+, since they cannot connect different sets S_i by Claim 3 and they cannot connect an S_i to $V\setminus S$ since $\delta(S)$ must contain exactly one edge of M and it already contains j. So exactly m edges of M connect S^+ to $S_1\cup\dots\cup S_m$, and one edge of M connects S^+ to $V\setminus S$. By Claim 5, no edge of M can connect two points in S^+. Hence $|S^+|=m+1$ as claimed.

Claim 7. *No edge connects a node in any S_i to a node in $V\setminus S_i$.*

For assume k is such an edge, and let M be a perfect matching containing k. Then the same counting as in the proof of Claim 6 yields that $|S^+|=m$, which is a contradiction.

By Claims 3, 6 and 7 we see that $G\setminus S^+$ has $m+1=|S^+|$ odd components. This is only possible if $|S^+|=1$, i.e., $S^+=S=\{u\}$. But this contradicts Claim 2. ∎

5. The matching rank of general graphs

Let $r(G)$ denote the maximum number of perfect matchings in G whose incidence vectors are linearly independent over $\mathbf{R}$. (We shall say simply that the perfect matchings are linearly independent.) It is clear that instead of linearly independent we could say affinely independent, since all incidence vectors of perfect matchings lie in the hyperplane $x(E)=|V|/2$, which does not go through the origin. It follows therefore that

$$r(G) = 1+\dim PM(G).$$

The main theorem of this paper is the following.

Theorem 5.1. *Let $G=(V, E)$ be a matching-covered graph. Then any brick decomposition of G results in $|E|-|V|+2-r(G)$ bricks.*

We had to be somewhat careful in formulating this result, since we have not yet proved that every brick decomposition of G results in the same number of bricks. Of course, the theorem implies this fact. Note that the theorem also provides an efficient way to determine $r(G)$: just carry out the brick decomposition of G in an arbitrary way. If we obtain $\beta(G)$ bricks then

$$r(G) = |E|-|V|+2-\beta(G).$$

Two special cases are worth pointing out. If G is bipartite then it has no bricks and so we obtain the following result (Naddef [11]—see Thm 1.2).

Corollary 5.2. *If G is a bipartite matching covered graph then*

$$r(G) = |E|-|V|+2.$$

If G is itself 3-connected bicritical, then $\beta(G)=1$ and hence we obtain

Corollary 5.3. *If G is a 3-connected bicritical graph then*

$$r(G) = |E|-|V|+1.$$

Before proving Theorem 5.1, we establish some lemmas which will also be used later on.

Lemma 5.4. *Let G be a graph and $G_1, \ldots, G_m$ the connected components of G. Then*

$$r(G) = r(G_1)+\ldots+r(G_m)-m+1.$$

Proof. It suffices to treat the case $m=2$; the general case follows then by induction. Let $M_1, \ldots, M_k$ be a basis of perfect matchings of G_1 and $M_1', \ldots, M_l'$ a basis of perfect matchings of G_2 (where $k=r(G_1)$, $l=r(G_2)$). Then $M_1\cup M_1'$, $M_1\cup M_2', \ldots$ $\ldots, M_1\cup M_l'$, $M_2\cup M_1', \ldots, M_k\cup M_1'$ is a basis of perfect matchings of $G_1\cup G_2$. The verification of this is left to the reader. (It can be proved in the same fashion as the next lemma.)

Lemma 5.5. *Let $J=\delta(S)$ be an odd cut of $G=(V, E)$ and let $\mathcal{M}$ be the set of all perfect matchings of G which contain exactly one edge of J. Assume that $\mathcal{M}\neq\emptyset$. Let $\bar{E}$ consist of those edges of G belonging to some $M\in\mathcal{M}$ and let $\bar{G}=(V, \bar{E})$. Then $r(\mathcal{M})=r(\bar{G}\times S)+r(\bar{G}\times(V\setminus S))-|J\cap\bar{E}|$.*

Proof. Let $\tilde{\mathcal{M}}$ and $\hat{\mathcal{M}}$ be sets of $r(\bar{G}\times S)$ and $r(\bar{G}\times(V\setminus S))$ affinely independent perfect matchings of $\bar{G}\times S$ and $\bar{G}\times(V\setminus S)$ respectively. Since for each $M\in\mathcal{M}$, the sets $M\setminus\gamma(V\setminus S)$ and $M\setminus\gamma(S)$ are perfect matchings of $\bar{G}\times S$ and $\bar{G}\times(V\setminus S)$ respectively, for each $j\in J\cap\bar{E}$ we can choose $\tilde{M}^j\in\tilde{\mathcal{M}}$ and $\hat{M}^j\in\hat{\mathcal{M}}$ such that $j\in\tilde{M}^j$ and $j\in\hat{M}^j$. Let $\tilde{\mathcal{M}}'$ and $\hat{\mathcal{M}}'$ be the sets of these $\tilde{M}^j$ and $\hat{M}^j$ respectively for $j\in\bar{E}\cap J$. We now define three subsets of $\mathcal{M}$.

$$\mathcal{M}_1\equiv\{\tilde{M}^j\cup\hat{M}^j\colon\ j\in J\cap\bar{E}\},$$

$$\mathcal{M}_2\equiv\{\tilde{M}\cup\hat{M}^j\colon\ \tilde{M}\in\tilde{\mathcal{M}}\setminus\tilde{\mathcal{M}}',\ \{j\}=\tilde{M}\cap(J\cap\bar{E})\},$$

$$\mathcal{M}_3\equiv\{\tilde{M}^j\cup\hat{M}\colon\ \hat{M}\in\hat{\mathcal{M}}\setminus\hat{\mathcal{M}}',\ \{j\}=\hat{M}\cap(J\cap\bar{E})\}.$$

We show that $\mathcal{M}_1\cup\mathcal{M}_2\cup\mathcal{M}_3$ is affinely independent. Suppose there exists a zero valued affine combination of $\mathcal{M}_1\cup\mathcal{M}_2\cup\mathcal{M}_3$, and suppose there exists $\tilde{M}\cup\hat{M}^j\in\mathcal{M}_2$ whose matching vector has a nonzero coefficient. Then restricting all matchings to $E(\bar{G}\times S)$ we see that these coefficients would give a nontrivial zero affine combination of members of $\tilde{\mathcal{M}}$, a contradiction. A similar argument holds for $\mathcal{M}_3$ so the coefficient corresponding to each member of $\mathcal{M}_2\cup\mathcal{M}_3$ has zero value. But since for $j\in J\setminus\bar{E}$, $\tilde{M}^j\setminus\hat{M}^j$ is the only member of $\mathcal{M}_1$ using j, we see that all coefficients are zero valued, and so $\mathcal{M}_1\cup\mathcal{M}_2\cup\mathcal{M}_3$ is affinely independent.

Now we show that $\mathcal{M}_1\cup\mathcal{M}_2\cup\mathcal{M}_3$ is an affine basis of $\mathcal{M}$. Let $M\in\mathcal{M}$ and let $\hat{j}$ be the edge of J in M. Since $\tilde{\mathcal{M}}$ was an affine basis of the perfect matchings of $\bar{G}\times S$, we can express $M\cap E(\bar{G}\times S)$ as an affine combination of members of $\tilde{\mathcal{M}}$. Let x be the vector obtained by taking that combination of the incidence vectors of the corresponding members of $\mathcal{M}_1\cup\mathcal{M}_2$. Then $x_j=0$ for $j\in J\setminus\{\hat{j}\}$ and $x_{\hat{j}}=1$. But for each $j\in J\cap E$, every member of $\mathcal{M}_1\cup\mathcal{M}_2$ that contains j equals $\hat{M}^j$ on $E(\bar{G}\times(V\setminus S))$. Therefore x restricted to $E(\bar{G}\times(V\setminus S))$ is the incidence vector of $\hat{M}^{\hat{j}}$. Analogously, we can express $M\cap E(\bar{G}\times(V\setminus S))$ as an affine combination of members of $\hat{\mathcal{M}}$. Let y be the vector obtained by taking that combination of the incidence vectors of the corresponding members of $\mathcal{M}_1\cup\mathcal{M}_3$. Then y restricted to $E(\bar{G}\times S))$ is the incidence vector of $\tilde{M}^{\hat{j}}$. Finally let z be the incidence vector of $\tilde{M}^{\hat{j}}\cup\hat{M}^{\hat{j}}\in\mathcal{M}_1$. Then the incidence vector of M equals $x+y-z$ and so M is an affine combination of members of $\mathcal{M}_1\cup\mathcal{M}_2\cup\mathcal{M}_3$.

Therefore $r(\mathcal{M})=|\mathcal{M}_1|+|\mathcal{M}_2|+|\mathcal{M}_3|=r(\bar{G}\times S)+r(\bar{G}\times(V\setminus S))-|J\cap\bar{E}|$ as asserted. ∎

Corollary 5.6. *Let $G=(V,E)$ be a matching-covered graph and let $J=\delta(S)$ be a strict cut in G. Then*

$$r(G)=r(G\times S)+r(G\times(V\setminus S))-|J|.\quad ∎$$

Let us remark that this result could be used to obtain a fairly simple inductive proof of Naddef's theorem. In fact, if G has no nontrivial strict cuts then no inequality (1.3) is tight and so dim $PM(G)$ is determined by the equations (1.2). If G has a non-trivial strict cut then we may apply Corollary 5.6 and induction.

Another simple lemma we need is the following.

Lemma 5.7. *Let $G=(V, E)$ be a graph and let $e \in E$ be an edge which is contained in a perfect matching in G. Let G' be the graph obtained from G by adding an edge e' parallel to e. Then $r(G')=r(G)+1$.*

Proof. G contains a perfect matching M which contains e. Let $M'=M \setminus e+e'$. Then clearly M' is affinely independent from the perfect matchings of G, and hence $r(G') \geqq r(G)+1$. Let M_1' be another perfect matching in G' containing e' an let $M_1= M_1' \setminus e'+e$. Then $M_1'=M_1 \setminus M \cup M'$, i.e., the incidence vector of M_1' is an affine combination of the incidence vectors of M' and perfect matchings of G. Hence $r(G') \leqq r(G)+1$. ∎

From these two lemmas it will be easy to deduce two similar but slightly more complicated formulas for the matching rank of matching-covered graphs.

Lemma 5.8. *Let $G=(V, E)$ be a matching-covered graph, $S \subseteq V(G)$, and suppose that the reduced lobes $G_1, \ldots, G_m$ of S in G are matching-covered and $m=|S|$. Then*

$$r(G) = \sum_{i=1}^{m} r(G_i)-2m+2.$$

(*Note that the condition of the theorem holds in particular if $S \in \mathcal{P}(G)$*).

Proof. Let $|V(G_1)|, \ldots, |V(G_k)|>2$ but $|V(G_{k+1})|=\ldots=|V(G_m)|=2$. We use induction on k. If $k=0$ then G is bipartite and for every i, $r(G_i)=|E(G_i)|$. So the formula says that

$$r(G) = \sum_{i=1}^{m} |E(G_i)|-2m+2 = |E|-|V|+2,$$

which is Corollary 5.2. In fact, after all our preparation, this relation is trivial: the equality system for $PM(G)$ consists only of the set of equations (1.2). Since G is bipartite, the rank of this system of equations is $|V|-1$, and so $r(G)=\dim PM(G)+1 =|E|-|V|+2$.

So suppose that $k>0$. Let $T=V(G_k) \setminus \{s\}$, where s is the pseudo-node coming from the shrinking of S.

Then $\delta(T)$ is a nontrivial strict cut and so by Corollary 5.6,

$$r(G) = r(G \times T)+r(G \times (V \setminus T))-|\delta(T)|.$$

Now by the induction hypothesis,

$$r(G \times T) = \sum_{i \neq k} r(G_i)+r(G \times T \times (V \setminus T))-2m+2 = \sum_{i \neq k} r(G_i)+|\delta(T)|-2m+2,$$

while $G \times (V \setminus T)=G_k$. Hence the identity in the Lemma follows. ∎

Lemma 5.9. *Let $G=(V, E)$ be a matching-covered graph, $\{u, v\}$ an articulation set of G and let $G_1, \ldots, G_m$ be the lobes of $\{u, v\}$. Let $G_i'=G_i+uv$, and let c be the multiplicity of uv in G. Then*

$$r(G) = \sum_{i=1}^{m} r(G_i')+c-m.$$

Proof. We prove by induction on m. Let $T=V(G_m)\setminus u$ and let a and b denote the numbers of edges connecting u to $T\setminus v$ and v to $V\setminus T\setminus u$, respectively. Then $\delta(T)$ is a strict cut and we may apply Corollary 5.6 again:

$$r(G) = r(G\times T)+r\big(G\times(V\setminus T)\big)-|\delta(T)|. \tag{5.1}$$

Now $G\times(V\setminus T)$ is just G'_m with $b+c-1$ further copies of uv added. Hence by Lemma 5.7,

$$r\big(G\times(V\setminus T)\big) = r(G'_m)+b+c-1.$$

Further, by the induction hypothesis,

$$r(G\times T) = \sum_{i=1}^{m-1} r(G'_1)+(c+a)-(m-1)$$

and clearly

$$|\delta(T)| = a+b+c.$$

Substituting these values in (5.1), the lemma follows. ∎

Proof of Theorem 5.1. This is now a matter of a fairly simple induction on $|V|$. Assume first that G is 3-connected bicritical. Then no constraint (1.1) is in the equality subsystem. By Theorem 4.7, no constraint (1.3) is in the equality subsystem. Therefore the equality subsystem consists of the set of equations (1.2) and so $\dim\big(PM(G)\big)=|E|-r(A)$, where A is the node-edge incidence matrix of G. Since G is nonbipartite and connected $r(A)=|V|$. Thus $r(G)=1+\dim\big(PM(G)\big)=|E|-|V|+1$.

Second, suppose that G is bicritical but not 3-connected. Let any decomposition of G start with an articulation set $\{u, v\}$, let $G_1, \ldots, G_m$ be the lobes of $\{u, v\}$ and $G'_i=G_i+uv$. Let c be the multiplicity of uv in G. Again, by definition,

$$\beta(G) = \sum_{i=1}^{m} \beta(G'_1)$$

and by the induction hypothesis

$$\beta(G'_i) = |E(G'_i)|-|V(G'_i)|+2-r(G'_i).$$

Hence

$$\beta(G) = \sum |E(G'_i)|-\sum |V(G'_i)|+2m-\sum r(G'_i) =$$

$$|E|-c+m-(|V|+2m-2)+2m-\sum r(G'_i)=|E|-c-|V|+2+m-\sum r(G'_i).$$

By Lemma 5.9,

$$\sum r(G'_i) = r(G)+m-c.$$

Hence the formula in the theorem follows.

Third, suppose that G is not bicritical. Let any decomposition of G start with a class $S\in\mathcal{P}(G)$, $m=|S|\geqq 2$. Let $G_1, \ldots, G_m$ be the reduced lobes of S. Then by definition, the number $\beta(G)$ of 3-connected bicritical constituents of G obtained in this procedure is

$$\beta(G) = \beta(G_1)+\ldots+\beta(G_m).$$

Further, by the induction hypothesis we know that

$$\beta(G_i) = |E(G_i)|-|V(G_i)|+2-r(G_i),$$

and so

$$\beta(G) = \sum \beta(G_i) = \sum |E(G_i)| - \sum |V(G_i)| + 2m - \sum r(G_i) = |E| - |V| + 2m - \sum r(G_i).$$

By Lemma 5.8,

$$\sum r(G_i) = r(G) + 2m - 2,$$

and so

$$\beta(G) = |E| - |V| + 2 - r(G)$$

as claimed. ∎

The value $\beta(G)$ is somewhat complicated to compute, therefore it may be of interest to deduce lower bounds on $r(G)$ (or, equivalently, upper bounds on $\beta(G)$ in terms of $|E|$ and $|V|$.

Lemma 5.10. For every bicritical graph $G=(V, E)$

$$\beta(G) \leqq |E| - \frac{3}{2}|V| + 1.$$

Proof. If G is 3-connected then $\beta(G)=1$, so the assertion is equivalent to the following:

$$|E| \geqq \frac{3}{2}|V|.$$

This is clear since every point has degree at least 3.

If G is not 3-connected then let $\{u, v\}$ be an articulation set, $G_1, \dots, G_m$ the lobes of $\{u, v\}$ and $G_i' = G_i + uv$. Then, using induction,

$$\beta(G) = \sum \beta(G_i') \leqq \sum \left(|E(G_i')| - \frac{3}{2}|V(G_i')| + 1\right)$$

$$\leqq |E| + m - \frac{3}{2}(|V| + 2m - 2) + m = |E| - \frac{3}{2}|V| - m + 3 \leqq |E| - \frac{3}{2}|V| + 1. \quad ∎$$

Corollary 5.11. *For every bicritical grap G,*

$$r(G) \geqq \frac{|V|}{2} + 1.$$

The graph in Figure 2 shows that this bound is sharp.

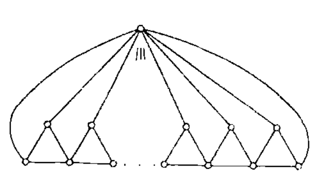

Fig. 2

By similar arguments one can show:

Lemma 5. 12. *If* $G=(V, E)$ *is a matching-covered graph then*

$$\beta(G) \leqq \frac{1}{2}(|E|-|V|).$$

Corollary 5.13. *For any matching-covered graph,*

$$r(G) \geqq \frac{1}{2}(|E|-|V|)+2.$$

One may use $r(G)$ as a lower bound on the number $\Phi(G)$ of perfect matchings in a graph. The lower bound in Corollary 5.13 was first proved by Lovász and Plummer ([8], Cor. 5.5.1). They also proved that $\Phi(G) \geqq \frac{1}{4}|V|+2$ and conjectured that $\Phi(G) \geqq \frac{1}{2}|V|+1$ for all bicritical graphs. Our Corollary 5.11 settles this conjecture in the affirmative. The proof method of Lovász and Plummer gives $\frac{1}{2}(|E|-|V|)+2$ perfect matchings which are linearly independent over any field. Our formula in Theorem 5.1 does not remain valid over fields of characteristic different from 0: the Petersen graph, which is 3-connected and bicritical, has 6 perfect matchings, which are linearly independent over **Q** but only 5 of them are linearly independent over $GF(2)$. To determine the number of perfect matchings linearly independent over $GF(2)$ (or over any other given finite field) is an outstanding open problem in matching theory. (See Naddef and Pulleyblank [13]).

We conclude this section with a characterization of those graphs $G=(V, E)$ for which $r(G)=|E|-|V|+1$ or, equivalently, for which $\beta(G)=1$. We call such a graph *rank-extreme.*

Let $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ be two graphs with $V_1 \cap V_2=\emptyset$. Let $v_1 \in V_1$ and $v_2 \in V_2$ have the same degree. A graph $G=(V, E)$ obtained from G_1 and G_2 by *splicing* v_1 and v_2 is defined as follows: $V:=V_1 \cup V_2 \setminus \{v_1, v_2\}$, $E:=(E_1 \setminus \delta(v_1)) \cup (E_2 \setminus \delta(v_2)) \cup J$ where J is obtained by pairing off the edges of $\delta(v_1)$ and $\delta(v_2)$ in some fashion, then for each pair (j_1, j_2), putting an edge joining the end of j_1 different from v_1 to the end of j_2 different from v_2. See Figure 3. Of course, different pairings to form J will in general result in different splicings.

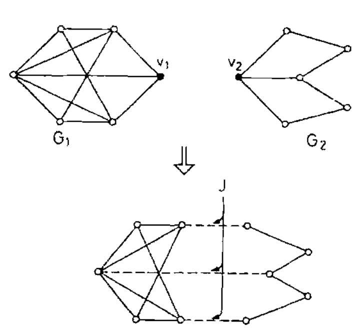

Fig. 3

Theorem 5.14. *A graph G is rank-extreme if and only if it can be obtained from a 3-connected bicritical graph H and a list $(G_v : v \in V(H))$ of matching-covered bipartite graphs by splicing each $v \in V(H)$ with an appropriate node $v' \in V(G_v)$.*

The proof follows easily from Theorem 5.1 and is omitted.

Corollary 5.15. *If G is a rank-extreme graph and $\delta(S)$ a strict cut in G, then one of $G \times S$ and $G \times (V \setminus S)$ is rank-extreme and the other is bipartite.*

6. The perfect matching polytope

In this section we give a minimal set of equations and inequalities which describe the perfect matching polytope $PM(G)$ of a graph $G=(V, E)$. Recall that this polytope is described by the following system:

(6.1) $\qquad x_j \geqq 0 \qquad$ for all $j \in E$,

(6.2) $\qquad x(\delta(i)) = 1 \qquad$ for all $i \in V$,

(6.3) $\qquad x(J) \geqq 1 \qquad$ for every non-trivial odd cut J of G.

We shall restrict ourselves to the case of matching-covered graphs, since edges which do not occur in perfect matchings can be omitted without causing any change in the perfect matching polytope, and if a graph is disconnected then its perfect matching polytope is just the Cartesian product of the perfect matching polytopes of its connected components.

We shall describe how to achieve the following tasks:

(a) Find a maximal system of linearly independent non-trivial strict odd cuts. These, together with the equations (6.2), will describe the affine hull of $PM(G)$. Note that since G is assumed to be matching-covered, no inequality in (6.1) is tight.

(b) Find those inequalities in (6.1) which are facet-inducing. Since the vertices of the face $\{x \in P : x_j = 0\}$ are just the perfect matchings of $G \setminus j$, this is tantamount to finding those edges j for which $r(G \setminus j) = r(G) - 1$.

(c) Find those inequalities in (6.3) which are facet-inducing.

(d) Describe which facet-inducing inequalities in (6.1) and (6.3) induce the same facet.

It will be convenient to start with the cases of bipartite and 3-connected bicritical graphs.

Theorem 6.1. *Let $G=(V, E)$ be a matching-covered bipartite graph with $|E| \geqq 2$. Then*

(a) *the affine hull of $PM(G)$ is described by the equations (6.2); these equations are not independent, but dropping any one of them from the system the rest is idependent,*

(b) *an inequality $x_j \geqq 0$ is facet-inducing if and only if either*
(B1) *$G \setminus j$ is matching.covered or*

(B2) *G has the following structure: it consists of vertex disjoint matching-covered bipartite subgraphs $G_1, \dots, G_m$ $(m \geqq 2)$ such that $V = V(G_1) \cup \dots \cup V(G_m)$ and m edges $e_1, \dots, e_{m-1}, e_m$ such that e_i connects G_i to G_{i+1} $(i=1, \dots, m-1)$ and e_m connects G_m to G_1, and $j \in \{e_1, \dots, e_m\}$. (Figure 4.)*

(c) *All inequalities (6.3) are inessential.*

(d) *The inequalities $x_j \geqq 0$ and $x_k \geqq 0$ induce the same facet if and only if G has the structure described in* (B2) *and $j, k \in \{e_1, \dots, e_m\}$.*

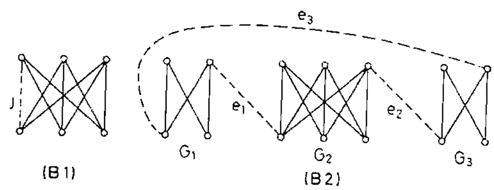

Fig. 4

Proof. (a) and (c) are well-known. To prove (b), let us assume first that $x_j \geqq 0$ induces a facet. If $G \setminus j$ is matching-covered we are done, so suppose that $G \setminus j$ is not matching-covered. Clearly G is 2-connected, and hence $G \setminus j$ is connected, so $G \setminus j$ must have edges not belonging to any perfect matching. Let E_0 be the set of these edges, and let $G_1, \dots, G_m$ be the connected components of $G \setminus E_0 \setminus j$. Since $G \setminus j$ has at least one perfect matching (just consider any perfect matching in G containing an edge adjacent to j), it follows that $G_1, \dots, G_m$ are matching-covered bipartite graphs and hence

$$r(G_i) = |E(G_i)| - |V(G_i)| + 2.$$

Now, by Lemma 5.4,

$$r(G \setminus j) = \sum_{i=1}^{m} r(G_i) - m + 1 = \sum |E(G_i)| - \sum |V(G_i)| + m + 1 =$$

$$= |E| - 1 - |E_0| - |V| + m + 1.$$

Since $G \setminus j$ is connected, $|E_0| \geqq m-1$ and thus

$$r(G \setminus j) \leqq |E| - |V| + 1.$$

We have equality here if and only if $|E_0| = m-1$, i.e., if all edges of E_0 are isthmuses in $G \setminus j$. It is easy to see that this implies that G has the structure (B2).

Conversely, if $G \setminus j$ is matching-covered then $r(G \setminus j) = |E| - 1 - |V| + 2 = r(G) - 1$ and so $x_j \geqq 0$ induces a facet.

Assume now that G has the structure (B2). Then as above,

$$r(G \setminus j) = |E| - |V| + 1 = r(G) - 1,$$

and so $x_j \geqq 0$ induces a facet.

To prove (d), assume first that $x_j \geqq 0$ and $x_k \geqq 0$ induce the same facet. Then every perfect matching vector with $x_j = 0$ must also satisfy $x_k = 0$, i.e., every perfect matching in $G \setminus j$ must avoid k. So $G \setminus j$ is not matching covered and by (b) it has structure (B2) and $k \in E_0$.

Conversely, assume that G has structure (B2) and $j, k \in \{e_1, \dots, e_m\}$. Then a perfect matching of G contains j if and only if it contains k. Thus $x_j = 0$ if and only if $x_k = 0$ for a vertex of $PM(G)$, which means that $x_j \geqq 0$ and $x_k \geqq 0$ induce the same facet. ∎

Theorem 6.2. *Let $G = (V, E)$ be a 3-connected bicritical graph. Then*

(a) *the affine hull of $PM(G)$ is described by the equations (6.2), which are independent.*

(b) *An inequality $x_j \geqq 0$ is facet-inducing if and only if either*

(B1) *$G \setminus j$ is rank-extreme or*

(B2) *there exists an edge $k \in E \setminus j$ such that $G \setminus j \setminus k$ is bipartite and matching-covered (Figure 5).*

(c) *An inequality $x(J) \geqq 1$ (where J is a non-trivial odd cut with shores S_1 and S_2) is facet-inducing if and only if either*

(C1) *$G \times S_i$ is rank-extreme for $i = 1, 2$,*

(C2) *for any one of the shores of J, say S_1, there exists an edge $e \in J \cup \gamma(S_1)$ such that $(G \setminus e) \times S_1$ is rank-extreme and $G \times S_2$ consists of a matching-covered bipartite graph together with the edge e joining two nodes in the color-class containing the pseudonode S_2; or*

(C3) *there exist edges $e_i \in J \cup \gamma(S_i)$ such that $G \times S_i$ consists of a matching-covered bipartite graph together with e_{3-i} joining two nodes in the color-class containing the pseudonode S_i. (Note that in case (C2), the edge e is facet-incuding as in (B1), and in case (C3), the edges e_1, e_2 are facet-inducing as in (B2); (Figure 6).*

(d) *Two of the above inequalities induce the same facet if and only if either*

(D1) *the graph has structure (B2) and the two inequalities are $x_j \geqq 0$ and $x_k \geqq 0$; or*

(D2) *the graph has structure (C2) and the two inequalities are $x(J) \geqq 1$ and $x_e \geqq 0$; or*

(D3) *the graph has structure (C3) and the two inequalities are $x(J) \geqq 1$ and $x_{e_1} \geqq 0$ or*

(D4) *the two inequalities are $x(J_1) \geqq 1$ and $x(J_2) \geqq 1$, none of the corresponding facets being edge-induced, and for appropriately chosen shores S_1 and S_2 of J_1 and J_2, respectively, $|S_1 \cap S_2|$ is odd, both $\delta(S_1 \cap S_2)$ and $\delta(S_1 \cup S_2)$ induce non-edge-induced facets, and $G \times (S_1 \cap S_2) \times (V \setminus S_1 \setminus S_2)$ is bipartite.*

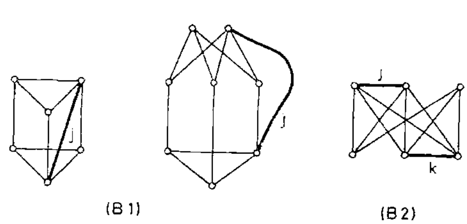

Fig. 5

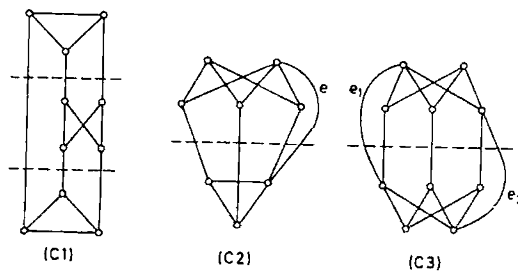

Fig. 6

Proof. (a) is essentially a re-statement of Corollary 5.3. To prove (b), assume first that $x_j \geqq 0$ induces a facet. If $G \setminus j$ is matching-covered then, by Theorem 5.1,

$$r(G \setminus j) = |E|-1-|V|+2-\beta(G \setminus j) = r(G)-\beta(G \setminus j),$$

whence $\beta(G \setminus j)=1$, i.e., $G \setminus j$ is rank-extreme. If $G \setminus j$ is not matching-covered then let E_0 be the set of edges of $G \setminus j$ not contained in any perfect matching and let $G_1, \ldots, G_m$ be the connected components of $G \setminus j \setminus E_0$. Then by Lemma 5.4 and Theorem 5.1,

$$r(G \setminus j) = r(G \setminus j \setminus E_0) = \sum_{i=1}^{m} r(G_i)-m+1 = \sum_{i=1}^{m} (|E(G_i)|-|V(G_i)|+2-\beta(G_i))-m+1$$

$$= |E|-|E_0|-1-|V|+2m-\sum_{i=1}^{m} \beta(G_i)-m+1.$$

Since $G \setminus j$ is 2-connected, $|E_0| \geqq m$ and thus

$$r(G \setminus j) \leqq |E|-|V|-\sum_{i=1}^{m} \beta(G_i) = r(G)-1-\sum_{i=1}^{m} \beta(G_i).$$

Since $x_j \geqq 0$ is assumed to be facet inducing, we obtain that $\beta(G_i)=0$, i.e., $G_1, \ldots, G_m$ are bipartite, and also that $|E_0|=m$.

We want to show that $m=1$. Suppose that $m \geqq 2$. Since G is 3-connected, every G_i is incident with at least 3 edges of $E_0 \cup \{j\}$. Hence $3m \leqq 2(m+1)$ and thus $m \leqq 2$. If $m=2$ then three edges of $E_0 \cup \{j\}$ connect the two bipartite matching-covered graphs G_1 and G_2, and so we may choose a color-class U_1 of G_1 and U_2 of G_2 such that U_1 and U_2 are not connected by any edge in G. But then $U_1 \cup U_2$ is an independent set of $\frac{1}{2}|V|$ points in G, and so for $x, y \in V \setminus U_1 \setminus U_2$, $G \setminus x \setminus y$ has no perfect matching, which is a contradiction. So $m=1$, and this proves that G does indeed have the structure (B2). The fact that if (B1) or (B2) holds then $x_j \geqq 0$ is facet-inducing follows similarly. Thus (b) is proved.

To prove (c), assume that $x(J) \geqq 1$ is facet-inducing, where $J=\delta(S)$ is an odd cut. Let $\mathcal{M}$ be the set of all perfect matchings of G which belong to this facet, i.e., which contain exactly one edge of J. Let $\bar{E}$ be the set of edges belonging to perfect matchings in $\mathcal{M}$ and let $\bar{G}=(V, \bar{E})$.

For every $j\in E\setminus\bar{E}$, we know that every perfect matching vector with $x(J)=1$ also satisfies $x_j=0$. Hence j is a facet inducing edge and it in fact induces the same facet as the cut J; and thus $x_j=0$ if and only if $x(J)=1$. Hence by part (b), $|E\setminus\bar{E}|\leqq 2$ and since G is 3-connected, we get that $\bar{G}$ is connected.

By Lemma 5.5,

$$r(\mathcal{M}) = r(\bar{G}\times S)+r(\bar{G}\times(V\setminus S))-|J\cap\bar{E}|, \tag{6.1}$$

and here, since J induces a facet,

$$r(\mathcal{M}) = r(G)-1 = |E|-|V|. \tag{6.2}$$

The graphs $\bar{G}\times S$ and $\bar{G}\times(V-S)$ are connected as $\bar{G}$ is connected and hence matching-covered. Thus by Theorem 5.1,

$$r(\bar{G}\times S) = |E(\bar{G}\times S)|-|V(\bar{G}\times S)|+2-\beta(\bar{G}\times S) = |E(\bar{G}\times S)|-|V-S|+1-\beta(\bar{G}\times S)$$

and

$$r(\bar{G}\times(V\setminus S)) = |E(\bar{G}\times(V\setminus S))|-|S|+1-\beta(\bar{G}\times(V\setminus S)).$$

So by (6.1) and (6.2)

$$|E|-|V| = |\bar{E}|-|V|+2-\beta(\bar{G}\times S)-\beta(\bar{G}\times(V\setminus S))$$

or

$$|E\setminus\bar{E}| = 2-\beta(\bar{G}\times S)-\beta(\bar{G}\times(V\setminus S)).$$

One further remark is needed. If $\bar{G}\times S$ is bipartite then let V_1 be its color class containing the point to which S is shrunk. Then at least one edge must connect points of V_1, since otherwise, $V_2=V(G\times S)-V_1$ would separate G into $|V_2|>1$ odd components, which is impossible as G is bicritical.

It is straightforward to see now that the case $\beta(\bar{G}\times S_1)=\beta(\bar{G}\times S_2)=1$ gives (C1), the case $\beta(\bar{G}\times S_1)=1$, $\beta(\bar{G}\times S_2)=0$ (or the other way around) gives (C2), and the case $\beta(\bar{G}\times S_1)=\beta(\bar{G}\times S_2)=0$ gives (C3).

The fact that in cases (C1), (C2) and (C3) the inequality $x(J)\geqq 0$ does indeed produce a facet follows easily from the calculations.

Finally, the proof of (d) in the cases (D1), (D2) and (D3) is essentially contained in the above arguments and details are omitted. It is somewhat more difficult to prove (d) in the case of two inequalities $x(J_1)\geqq 1$ and $x(J_2)\geqq 1$, which induce facets not induced by edges. Suppose first that $x(J_1)\geqq 1$ and $x(J_2)\geqq 1$ induce the same facet. Choose shores S_1 of J_1 and S_2 of J_2 such that $|S_1\cap S_2|$ is odd. We claim that $J'=\delta(S_1\cup S_2)$ and $J''=\delta(S_1\cap S_2)$ are odd cuts which also induce the same facet. For, let M be any perfect matching on the facet induced by J_1, i.e., let $|M\cap J_1|=1$. Then we also have $|M\cap J_2|=1$. Further, $|M\cap J'|\geqq 1$ and $|M\cap J''|\geqq 1$, since J' and J'' are odd cuts. But we have, by simple counting,

$$|M\cap J_1|+|M\cap J_2| \geqq |M\cap J'|+|M\cap J''|, \tag{6.3}$$

whence $|M\cap J'|=|M\cap J''|=1$. So every vertex of the facet induced by $x(J_1)\geqq 1$ satisfies $x(J')=1$, and thus J' is either strict or it induces the same facet as J_1. A similar conclusion holds for J''.

Next we show that J' and J'' cannot be strict. To this end, we first prove that no edge of G connects $S_1 \setminus S_2$ to $S_2 \setminus S_1$. Assume indirectly that e is such an edge. Since the facet $x(J_1) \geqq 1$ is not edge-induced, there must be a perfect matching M such that $|M \cap J_1| = 1$ and $e \in M$. Since J_2 induces the same facet, we have $|M \cap J_2| = 1$. But then the same counting as in the proof of (6.3) leads to a contradiction.

It follows now that it cannot happen that both J' and J'' are strict, since then by Theorem 4.7, they would have to be stars, i.e., $|S_1 \cap S_2| = 1$ and $|V \setminus S_1 \setminus S_2| = 1$. But this contradicts the hypothesis that G is 3-connected. Further, if one, say J'', is strict but the other, J', is not, then J' induces a non-edge-induced facet (the same facet as J_1 and J_2), and so $G_1 = G \times (V \setminus S_1 \setminus S_2)$ is rank-extreme by (b). Let $S_1 \cap S_2 = \{u\}$, and let v be the pseudonode corresponding to $V \setminus S_1 \setminus S_2$ in G_1. Let H_1 and H_2 be the subgraphs of G_1 induced by $(S_1 \setminus S_2) \cup \{u, v\}$ and $(S_2 \setminus S_1) \cup \{u, v\}$, and let $H_i' = H_i + uv$ for $i = 1, 2$. Then, up to the multiplicity of the edge uv, H_1' is just $G_1 \times S_2$ and H_2' is just $G_1 \times S_1$. Hence H_1' and H_2' are rank-extreme and so non-bipartite. But then $\beta(G_1) \geqq 2$, which is a contradiction with the fact that G_1 is rank-extreme. So J' and J'' are non-trivial, and hence non-strict cuts, and so they do indeed induce the same facet as J_1 and J_2.

Now J' is an odd cut in $G \times (S_1 \cap S_2)$, and since J' and J'' induce the same facet, J' must be a strict cut of $G \times (S_1 \cap S_2)$. But $G \times (S_1 \cap S_2)$ is rank-extreme by (b) and so one of $G \times (S_1 \cap S_2) \times (S_1 \triangle S_2)$ and $G \times (S_1 \cap S_2) \times (V \setminus S_1 \setminus S_2)$ must be bipartite while the other is rank extreme. But we know that $G \times (S_1 \cap S_2) \times (S_1 \triangle S_2) = G \times (S_1 \cup S_2)$ is rank-extreme, and hence $G \times (S_1 \cap S_2) \times (V \setminus S_1 \setminus S_2)$ is bipartite. This proves the necessity of the condition described in (D4). The sufficiency follows by a straightforward computation of the ranks. ∎

We now describe, in an inductive way, a minimal set of equations and inequalities which determine $PM(G)$ for a general matching-covered graph G. More precisely, we shall construct three things.

(α) If G is bipartite, we already know that all but one equation of (6.2) form a minimal set of equations determining $PM(G)$. If G is non-bipartite, we take all equations of (6.2) and some equations of $x(J) = 1$, where J ranges over a laminar family $F(G)$ of odd cuts with the odd cycle property.

(β) The facet-inducing edges are best described along with the information telling which of them induce the same facet. Thus we shall consider a coloration $\varphi(G)$ of a subset of $E(G)$ for every matching-covered graph G, such that $j \in E(G)$ induces a facet if and only if it is colored, and two edges induce the same facet if and only if they have the same color.

(γ) Further, we shall describe a minimum list $L(G)$ of odd cuts which induce all non-edge-induced facets of $PM(G)$.

For 3-connected bicritical graphs and matching-covered bipartite graphs, $F(G)$, $\varphi(G)$ and $L(G)$ are easily determined by Theorems 6.1 and 6.2. The next theorem tells us how to proceed to general matching-covered graphs.

Theorem 6.3. *Let G be a matching-covered non-bipartite graph and J a non-trivial strict cut in G with shores S_1 and S_2. Then $F(G)$, $\varphi(G)$ and $L(G)$ can be constructed as follows.*

(α) *If both $G \times S_1$ and $G \times S_2$ are non-bipartite, then let $F(G) = F(G \times S_1) \cup F(G \times S_2) \cup \{J\}$. If $G \times S_1$ is bipartite (say) but $G \times S_2$ is not, then $F(G) = F(G \times S_2)$. (It cannot happen that both $G \times S_1$ and $G \times S_2$ are bipartite.)*

(β) *First make all colors in $\varphi(G_1)$ and $\varphi(G_2)$ distinct. Delete all colors which occur as the color of an edge $e \in J$ in $\varphi(G_i)$ such that e is not colored in $\varphi(G_{3-i})$. Identify a color in $\varphi(G_1)$ with a color in $\varphi(G_2)$ if they occur as colors of the same edge $e \in J$.*
(γ) *Take $L(G)=L(G\times S_1)\cup L(G\times S_2)$.*

Proof. (α), (β), and the part of (γ) saying that the cuts in $L(G\times S_1)\cup L(G\times S_2)$ induce distinct, non-edge-induced facets of $PM(G)$ can be proved by straightforward computation based on Lemma 5.5, and we do not go into the details. To prove that every non-edge-induced facet of $PM(G)$ is induced by a cut in $L(G\times S_1)\cup L(G\times S_2)$ is slightly trickier. Let J' be a cut in G inducing a non-edge-induced facet with shores T_1 and T_2 labelled so that $|T_1\cap S_1|$ is odd.

Claim. *No edge connects $T_1\cap S_2$ to $T_2\cap S_1$.*

Assume that j is such an edge, and let M be any perfect matching containing j. Then since $|T_1\cap S_2|$ is even, M must contain another edge k from $\delta(T_1\cap S_2)$. Since $J=\delta(S_2)$ is strict, k must have its other endnode in $T_2\cap S_2$, and so $k\in\delta(T_1)=J'$. This means that no perfect matching on the facet induced by J contains j, i.e., j induces the same facet as J, which contradicts the hypothesis that the facet $x(J)=1$ is not edge-induced. Thus the claim is proved.

It follows just like in the proof of Theorem 6.3(d) that every perfect matching on the face induced by J' satisfies $x(\delta(T_1\cap S_1))=x(\delta(T_2\cap S_2))=1$. Hence the odd cuts $\delta(T_1\cap S_1)$ and $\delta(T_2\cap S_2)$ are either strict or induce the same facet as J'. But it cannot happen that both are strict, since then for every perfect matching M, by the Claim, it would follow that

$$|M\cap J'| = |M\cap\delta(S_1\cap T_1)|+|M\cap\delta(S_2\cap T_2)|-|M\cap J| = 1,$$

whence J' is strict, which is, however, not the case. ∎

References

[1] J. Edmonds, Maximum matching and a polyhedron with 0—1 vertices, *Journal of Research of the National Bureau of Standards* **69B** (1965) 125—130.

[2] J. Edmonds, Paths, trees and flowers, *Canadian Journal of Mathematics* **17** (1965) 449—467.

[3] J. Edmonds, Matroid intersection, *Annals of Discrete Mathematics* **4** (1979) 39—49.

[4] J. Edmonds, Systems of distinct representatives and linear algebra, *Journal of Research of the National Bureau of Standards* **71B** (1967) 241—245.

[5] T. Gallai, Maximale Systeme unabhängiger Kanten, *Mat. Kut. Int. Közl.* **9** (1964) 353—395.

[6] M. Grötschel, L. Lovász and A. Schrijver, The ellipsoid method and its consequences for combinatorial optimization, *Combinatorica* **1**(2) (1981) 169—197.

[7] A. Kotzig, Ein Beitrag zur Theorie der endlichen Graphen mit linearen Faktoren I—II—III (in Slovak with a Germany summary), *Math. Fyz. Casopis*, **9** (1959) pp. 83—91, 136—159, **10** (1960) 205—215.

[8] L. Lovász, On the structure of factorizable graphs, *Acta Math. Acad. Sci. Hung.* **23** (1972) 179—195.

[9] L. Lovász and M. D. Plummer, On bicritical graphs, *Infinite and Finite Sets*, Colloqu. Math. Soc. J. Bolyai **10**, Budapest, (A. Hajnal, R. Rado and V. T. Sós et al eds) (1975) 1051—1079.

[10] G. Minty, On maximal independent sets of vertices in claw-free graphs, *Journal of Combinatorial Theory, Series B* **28** (1980) 284—304.

[11] D. Naddef, Rank of maximum matchings of a graph, *Mathematical Programming* **22** (1982) 52—70.

[12] D. Naddef and W. R. Pulleyblank, Matchings in regular graphs, *Discrete Mathematics* **34** (1981) 283—290.
[13] D. Naddef and W. R. Pulleyblank, On GF_2 rank and ear decomposition of elementary graphs, *Annals of Discrete Mathematics* **16** (1982) 285—304.
[14] W. R. Pulleyblank, The matching rank of Halin graphs, *Report No. 80165-OR, Inst. für Operations Research, Universität Bonn (1980).*
[15] W. R. Pulleyblank and J. Edmonds, Facets of 1-matching polyhedra, *Hypergraph Seminar*, (C. Berge and D. K. Ray-Chaudhuri eds), *Springer Verlag* (1974), 214—242.

J. Edmonds, W. R. Pulleyblank

Dept. of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario, N2L 3G1, Canada

L. Lovász

Mathemathical Institute
of L. Eötvös University
Budapest, 1088, Hungary